

Ground state pressure and energy density of an interacting homogeneous Bose gas in two dimensions

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Abstract. We consider an interacting homogeneous Bose gas at zero temperature in two spatial dimensions. The properties of the system can be calculated as an expansion in powers of g , where g is the coupling constant. We calculate the ground state pressure and the ground state energy density to second order in the quantum loop expansion. The renormalization group is used to sum up leading and subleading logarithms from all orders in perturbation theory. In the dilute limit, the renormalization group improved pressure and energy density are expansions in powers of the T^{2B} and $T^{2B} \ln(T^{2B})$, respectively, where T^{2B} is the two-body T -matrix.

PACS. 03.75.Fi Phase coherent atomic ensembles; quantum condensation phenomena – 05.45.Ac Low-dimensional chaos

1 Introduction

The remarkable achievement of Bose-Einstein condensation (BEC) of alkali atoms in harmonic traps [1–3] has created an enormous interest in the properties of dilute Bose gases (for a review, see *e.g.* Ref. [4] and references therein).

The homogeneous Bose gas in three dimensions has been studied in great detail over the past 50 years (for a review, see *e.g.* [5]). At zero temperature, the quantum loop expansion is essentially an expansion in powers of $\sqrt{\rho a^3}$, where a is the two-body s -wave scattering length and ρ is the density. Lee and Yang [6] were the first to calculate the leading quantum correction to the energy density. Part of the second quantum correction to the energy density was obtained by Wu, by Hugenholtz and Pines, and by Sawada [7]. Only recently has a complete two-loop result been obtained by Braaten and Nieto [8]. The result depends, in addition to the scattering length, also on the scattering amplitude for $3 \rightarrow 3$ scattering.

The homogeneous Bose gas in two dimensions has also been studied extensively. The chemical potential and ground state energy density of a two-dimensional homogeneous Bose gas were first calculated by Schick [10]. By summing up ladder diagrams, he showed that in the dilute limit, the leading order results for the energy density and chemical potential are proportional to the two-body T -matrix, $T^{2B} \sim [\ln(\rho a^2)]^{-1}$, where ρ is the density and a is the range of the interaction. Corrections to these results have been considered by several authors [11–16].

A formal proof of the result by Schick was given by Lieb and Yngvason [17,18]. Very recently, the two-dimensional Bose gas has been studied by Bernardet *et al.* [19] using lattice simulations.

A Bose-Einstein condensate in a two-dimensional homogeneous Bose gas only exists at $T = 0$. At finite temperature, phase fluctuations destroy the condensate. This is reflected in the Mermin-Wagner theorem [20,21], which states that there is no spontaneous breakdown of a continuous symmetry in a homogeneous system in two dimensions at finite temperature. The physics of two-dimensional homogeneous Bose gases is nonetheless very interesting. A dilute homogeneous Bose gas in two dimensions is expected to undergo a phase transition at finite temperature, which is the Kosterlitz-Thouless transition [22]. Below the critical temperature, the gas is superfluid but has only algebraic long-range order. This topological phase transition is not characterized by a local order parameter, but by the unbinding of vortex pairs and the destruction of superfluidity. The superfluid phase is characterized by the existence of a quasicondensate [12] which roughly speaking corresponds to a condensate with a fluctuating phase. The existence of a quasicondensate has been observed in spin-polarized hydrogen adsorbed on a superfluid ^4He surface by Safonov *et al.* [23].

In the present paper, we reconsider the homogeneous Bose gas in two dimensions at zero temperature. We calculate the pressure and energy density of the ground state to second order in the quantum loop expansion. We also apply the renormalization group to sum up leading and subleading logarithms from all orders of perturbation theory. In the dilute limit, the renormalization group improved

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pressure and energy density are essentially expansions in powers of T^{2B} and $T^{2B} \ln(T^{2B})$, respectively.

The paper is organized as follows. In Section 2, we discuss the perturbative framework developed in reference [8] to calculate the ground state properties of a homogeneous Bose gas. In Section 3, we calculate the ground state pressure to two-loop order. In Section 4, we calculate the ground state energy density through two loops. In Section 5, we apply the renormalization group to sum up leading and subleading logarithms from all orders in perturbation theory. Finally, we summarize in Section 6. Computational details are included in an appendix.

2 Perturbative framework

In this section, we discuss the perturbative framework set up in reference [8] to calculate the effects on the ground state from quantum fluctuations around the mean field.

The action is

$$S = \int dt \left\{ \int d^2x \psi^*(\mathbf{x}, t) \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 + \mu \right] \psi(\mathbf{x}, t) - \frac{1}{2} \int d^2x \int d^2x' \left[\psi^*(\mathbf{x}, t) \psi^*(\mathbf{x}', t) V_0(\mathbf{x} - \mathbf{x}') \times \psi(\mathbf{x}, t) \psi(\mathbf{x}', t) \right] \right\}. \quad (1)$$

$\psi^*(\mathbf{x}, t)$ is a complex field operator that creates a boson at the position \mathbf{x} , μ is the chemical potential, and $V_0(\mathbf{x})$ is the two-body potential. In the following, we set $\hbar = 2m = 1$. Factors of \hbar and $2m$ can be reinserted using dimensional analysis.

The action equation (1) is symmetric under a phase transformation

$$\psi(\mathbf{x}, t) \rightarrow e^{i\alpha} \psi(\mathbf{x}, t). \quad (2)$$

The $U(1)$ -symmetry ensures that the density ρ and current density \mathbf{j} satisfy the continuity equation

$$\dot{\rho} + \nabla \cdot \mathbf{j} = 0. \quad (3)$$

In the ground state, the current density \mathbf{j} vanishes identically and the condensate has a constant phase. The $U(1)$ -symmetry can then be used to make the condensate real everywhere.

If the interatomic potential $V_0(\mathbf{x})$ is short range, it can be mimicked by local interactions. If the energies are low enough, the scattering amplitude can be approximated by s -wave scattering and the action equation (4) can be approximated by [9]

$$S = \int dt \int d^2x \psi^* \left[i \frac{\partial}{\partial t} + \nabla^2 + \mu \right] \psi - \frac{1}{2} g (\psi^* \psi)^2. \quad (4)$$

Here, g is a coupling constant that must be tuned to reproduce some low-energy observable of the true potential $V_0(\mathbf{x})$.

The quantum field theory defined by the action equation (4) has ultraviolet divergences that must be removed by renormalization of μ and g . There is also an ultraviolet divergence in the expression for the density ρ . This divergence can be removed by adding a counterterm $\delta\rho$. Alternatively, one can eliminate the divergences associated with μ and ρ by a normal-ordering prescription of the fields in equation (4). The coupling constant is renormalized in the usual way by replacing the bare coupling with the physical one.

If we use a simple momentum cutoff M to cut off the ultraviolet divergences in the loop integrals, there will be terms proportional to M^p , where p is a positive integer. There are also terms that are proportional to $\ln(M)$. The coefficients of the power divergences depend on the regularization method and are therefore artifacts of the regulator. On the other hand, the coefficients of $\ln(M)$ are independent of the regulator and they therefore represent real physics. In this paper, we use dimensional regularization to regulate both infrared and ultraviolet divergences. In dimensional regularization, one calculates the loop integrals in $d = 2 - 2\epsilon$ dimensions for values of ϵ where the integrals converge. One then analytically continues back to $d = 2$ dimensions. With dimensional regularization, an arbitrary renormalization scale M is introduced. This scale can be identified with the simple momentum cutoff mentioned above. An advantage of dimensional regularization is that it automatically sets power divergences to zero, while logarithmic divergences show up as poles in ϵ . In two dimensions, the one-loop counterterms for the chemical potential μ and the density ρ are quadratic ultraviolet divergences, while the one-loop counterterm for the coupling constant g is a logarithmic ultraviolet divergence. At the two-loop level, the counterterms for the chemical potential and the density are also quadratic divergences. The counterterm for the coupling constant is a double logarithmic divergence.

We next parameterize the quantum field ψ in terms of a time-independent condensate v and a quantum fluctuating field $\tilde{\psi}$:

$$\psi = v + \tilde{\psi}. \quad (5)$$

The fluctuating field can be written in terms of two real fields:

$$\tilde{\psi} = \frac{1}{\sqrt{2}} (\psi_1 + i\psi_2). \quad (6)$$

Substituting equation (6) into equation (4), the action can be decomposed into three terms

$$S[\psi] = S[v] + S_{\text{free}}[\psi_1, \psi_2] + S_{\text{int}}[v, \psi_1, \psi_2]. \quad (7)$$

$S[v]$ is the classical action

$$S[v] = \int dt \int d^2x \left[\mu v^2 - \frac{1}{2} g v^4 \right], \quad (8)$$

while the free part of the action is

$$S_{\text{free}}[\psi_1, \psi_2] = \int dt \int d^2x \left[\frac{1}{2} (\dot{\psi}_1 \psi_2 - \psi_1 \dot{\psi}_2) + \frac{1}{2} \psi_1 (\nabla^2 + X) \psi_1 + \frac{1}{2} \psi_2 (\nabla^2 + Y) \psi_2 \right]. \quad (9)$$

The interaction part of the action is

$$S_{\text{int}}[v, \psi_1, \psi_2] = \int dt \int d^2x \left[\sqrt{2T} \psi_1 + \frac{1}{\sqrt{2}} Z \psi_1 (\psi_1^2 + \psi_2^2) - \frac{1}{8} g (\psi_1^2 + \psi_2^2)^2 \right]. \quad (10)$$

The sources in equation (10) are

$$T = [\mu - gv^2] v, \quad (11)$$

$$X = [\mu - 3gv^2], \quad (12)$$

$$Y = [\mu - gv^2], \quad (13)$$

$$Z = -gv. \quad (14)$$

The propagator that corresponds to the free action $S_{\text{free}}[\psi_1, \psi_2]$ in equation (9) is

$$D(\omega, p) = \frac{i}{\omega^2 - \epsilon^2(p) + i\epsilon} \begin{pmatrix} p^2 - Y & -i\omega \\ i\omega & p^2 - X \end{pmatrix}. \quad (15)$$

Here \mathbf{p} is the wavevector, ω is the frequency, and $\epsilon(p)$ is the dispersion relation:

$$\epsilon(p) = \sqrt{(p^2 - X)(p^2 - Y)}. \quad (16)$$

The value of the condensate v_0 that minimizes the classical action is given by the equation $T = 0$. Both the propagator equation (15) and the dispersion relation equation (16) greatly simplify at v_0 , since $Y = 0$ here. The dispersion relation then becomes gapless, which reflects the spontaneous breakdown of the $U(1)$ -symmetry (Goldstone's theorem). The dispersion relation is linear for small wavevectors and is quadratic for large wavevectors, which is that of a free nonrelativistic particle. The propagator is defined with an $i\epsilon$ prescription in the usual way.

The partition function \mathcal{Z} can be expressed as a path integral

$$\mathcal{Z} = \int \mathcal{D}\psi_1 \mathcal{D}\psi_2 e^{iS[\psi_1, \psi_2]}. \quad (17)$$

All the thermodynamic observables can be derived from \mathcal{Z} . For instance, the free energy density \mathcal{F} is given by

$$\mathcal{F}(\mu) = i \frac{\ln \mathcal{Z}}{VT}, \quad (18)$$

where VT is the spacetime volume of the system.

The density ρ is given by the expectation value $\langle \psi^\dagger \psi \rangle$ in the ground state. It can therefore be expressed as

$$\rho(\mu) = - \frac{\partial \mathcal{F}(\mu)}{\partial \mu}. \quad (19)$$

The energy density \mathcal{E} is given by the Legendre transform of the the free energy density

$$\mathcal{E}(\rho) = \mathcal{F}(\mu) + \rho\mu. \quad (20)$$

At this point it is convenient to introduce the thermodynamic potential $\Omega(\mu, v)$. The thermodynamic potential is given by all one-particle irreducible vacuum graphs and can be expanded in the number of loops

$$\Omega(\mu, v) = \Omega_0(\mu, v) + \Omega_1(\mu, v) + \Omega_2(\mu, v) + \dots, \quad (21)$$

where the subscript n indicates the contribution from the n th order in the loop expansion. The free energy \mathcal{F} is given by all connected vacuum graphs and is independent of the condensate v . If we evaluate Ω at a value of the condensate that satisfies the condition

$$\bar{v} = \langle \psi \rangle, \quad (22)$$

it can be shown that all the one-particle reducible graphs (those that can be disconnected by cutting a single line) vanish. We then have

$$\mathcal{F}(\mu) = \Omega_0(\mu, \bar{v}) + \Omega_1(\mu, \bar{v}) + \Omega_2(\mu, \bar{v}) + \dots \quad (23)$$

The condition equation (22) is equivalent to

$$\frac{\partial \Omega}{\partial v} = 0. \quad (24)$$

The loop expansion equation (23) does not coincide with the expansion in powers quantum corrections. To obtain the expansion in powers of quantum corrections, we must expand the condensate \bar{v} about the classical minimum v_0 :

$$\bar{v} = v_0 + v_1 + v_2 + \dots \quad (25)$$

By substituting equation (21) into equation (24), we obtain the first quantum correction v_1 to the classical minimum:

$$v_1 = - \frac{\partial \Omega_1(\mu, v)}{\partial v} \Big|_{v=v_0} / \frac{\partial^2 \Omega_0(\mu, v)}{\partial v^2} \Big|_{v=v_0}. \quad (26)$$

This first quantum correction to the free energy density is

$$\mathcal{F}_1(\mu) = \Omega_1(\mu, v_0), \quad (27)$$

and the second quantum correction to the free energy density is

$$\mathcal{F}_2(\mu) = \Omega_2(\mu, v_0) + v_1 \frac{\partial \Omega_1(\mu, v)}{\partial v} \Big|_{v=v_0} + \frac{1}{2} v_1^2 \frac{\partial^2 \Omega_0(\mu, v)}{\partial v^2} \Big|_{v=v_0}. \quad (28)$$

3 Pressure to two loops

In this section, we calculate the pressure as a function of the chemical potential μ and the renormalized coupling g to two loops.

3.1 Mean-field free energy

The thermodynamic potential in the mean-field approximation is

$$\Omega_0(\mu, v) = -\mu v^2 + \frac{1}{2} g v^4. \quad (29)$$

The mean-field free energy is given by the classical thermodynamic potential evaluated at the classical minimum v_0 :

$$\mathcal{F}_0(\mu) = -\frac{\mu^2}{2g}. \quad (30)$$

3.2 One-loop free energy

The one-loop contribution to the free energy is

$$\mathcal{F}_1(\mu) = \mathcal{F}_{1a}(\mu) + \Delta_1 \mathcal{F}, \quad (31)$$

where

$$\mathcal{F}_{1a}(\mu) = \frac{i}{2} \int \frac{d\omega}{2\pi} \int \frac{d^d p}{(2\pi)^d} \ln \det D^{-1}, \quad (32)$$

and $\Delta_1 \mathcal{F}$ is the one-loop counterterm

$$\Delta_1 \mathcal{F} = -\frac{\mu}{g} \Delta_1 \mu + \frac{\mu^2}{2g^2} \Delta_1 g. \quad (33)$$

The propagator $D(\omega, p)$ is evaluated at the classical minimum, where $Y = 0$. By integrating over ω , equation (32) becomes

$$\begin{aligned} \mathcal{F}_{1a}(\mu) &= \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \epsilon(p) \\ &= \frac{1}{2} I_{0,-1}(2\mu), \end{aligned} \quad (34)$$

where $I_{m,n}$ is defined in Appendix A. Using equation (A.9) in Appendix A, we obtain

$$\mathcal{F}_{1a}(\mu) = -\frac{\mu^2}{16\pi} \left[\frac{1}{\epsilon} - L - \frac{1}{2} + \mathcal{O}(\epsilon) \right], \quad (35)$$

where $L = \ln(\mu/2M^2)$. The counterterms at one-loop are [24]:

$$\Delta_1 \mu = 0, \quad (36)$$

$$\Delta_1 g = \frac{g^2}{8\pi\epsilon}. \quad (37)$$

Adding equations (33, 35), we obtain the one-loop contribution to the free energy

$$\mathcal{F}_1(\mu) = \frac{\mu^2}{16\pi} \left[L + \frac{1}{2} \right]. \quad (38)$$

Note that $\mathcal{F}_1(\mu)$ is independent of the coupling constant g . Adding equations (30, 39), we obtain the one-loop approximation to the free energy:

$$\mathcal{F}_{0+1}(\mu) = -\frac{\mu^2}{g} + \frac{\mu^2}{16\pi} \left[L + \frac{1}{2} \right], \quad (39)$$

where the coupling constant is evaluated at the scale M , $g = g(M)$.

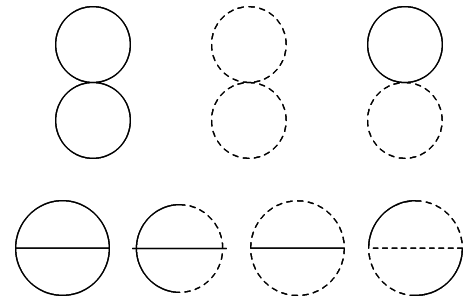


Fig. 1. Two-loop vacuum graphs contributing to the free energy \mathcal{F} .

3.3 Two-loop free energy

The two-loop contribution to the thermodynamic potential is given by the one-particle irreducible graphs shown in Figure 1. The solid line denotes the diagonal propagator for ψ_1 and the dashed line denotes the diagonal propagator for ψ_2 . The off-diagonal propagators for ψ_1 and ψ_2 are denoted by a solid-dashed and dashed-solid line, respectively. The expression for the diagrams is

$$\Omega_2(\mu, v) = \frac{g\mu}{8} J + \frac{g}{32} [3I_{1,1}^2 + 2I_{-1,-1}I_{1,1} + 3I_{-1,-1}^2], \quad (40)$$

where

$$J = 6J_{0,0,1} - J_{-1,-1,1} - 3J_{1,1,1} - 2J_{-1,0,0}, \quad (41)$$

and integrals $J_{l,m,n}$ are defined in Appendix A. The first correction v_1 is given by equation (26):

$$v_1 = -\frac{gv_0}{8\mu} [3I_{1,1} + I_{-1,-1}]. \quad (42)$$

The two-loop contribution to the free energy is then

$$\mathcal{F}_2(\mu) = \frac{g\mu}{8} J + \frac{g}{16} [I_{-1,-1}^2 - 2I_{-1,-1}I_{1,1} - 3I_{1,1}^2] + \Delta_2 \mathcal{F}. \quad (43)$$

The counterterm $\Delta_2 \mathcal{F}$ is

$$\begin{aligned} \Delta_2 \mathcal{F} &= \frac{1}{2} I_{1,1} \Delta_1 \mu - \frac{\mu}{g} \Delta_2 \mu + \frac{\mu^2}{2g^2} \Delta_2 g \\ &\quad - \frac{1}{2g} (\Delta_1 \mu)^2 + \frac{\mu}{g^2} \Delta_1 \mu \Delta_1 g - \frac{\mu^2}{2g^3} \Delta_1 g^2. \end{aligned} \quad (44)$$

At the two-loop level, the counterterms are given by

$$\Delta_2 \mu = 0, \quad (45)$$

$$\Delta_2 g = \frac{g^3}{64\pi^2 \epsilon^2}. \quad (46)$$

The integrals $J_{l,m,n}$ are ultraviolet divergent. After subtracting the divergent part, the remainder must be calculated numerically. We evaluate the expression in Appendix A. The result is

$$J = -2I_{0,1} [I_{1,1} + I_{-1,-1}] + J_{\text{num}}, \quad (47)$$

where $J_{\text{num}} = -3.52 \times 10^{-5} \mu$. The final result for the two-loop contribution to the free energy is

$$\mathcal{F}_2(\mu) = -\frac{g\mu^2}{64\pi^2} [1 + C]. \quad (48)$$

Here $C = 2.78 \times 10^{-3}$. Adding equations (30, 39), and (48), we obtain our final result for the free energy to second order in the quantum loop expansion

$$\mathcal{F}_{0+1+2}(\mu) = -\frac{\mu^2}{2g} + \frac{\mu^2}{16\pi} \left[L + \frac{1}{2} \right] - \frac{g\mu^2}{64\pi^2} [1 + C], \quad (49)$$

3.4 Two-loop pressure

The pressure \mathcal{P} is given by $-\mathcal{F}$. The pressure through two loops is given by minus the sum of equations (30, 39) and (48):

$$\mathcal{P}_{0+1+2}(\mu) = \frac{\mu^2}{2g} - \frac{\mu^2}{16\pi} \left[L + \frac{1}{2} \right] + \frac{g\mu^2}{64\pi^2} [1 + C]. \quad (50)$$

The coupling constant g in equation (4) satisfies

$$M \frac{d}{dM} g = \beta(g), \quad (51)$$

where the β -function is a polynomial in g . Normally, the β -functions are known only up to a certain order in the quantum loop expansion. In the present case, the one-loop result for the β -function is exact and $\beta(g) = g^2/4\pi$ [24]. From equation (51), one can easily check that our result equation (50) for the two-loop pressure is independent of the scale M up to correction of order g^2 .

4 Energy density to two loops

In this section, we derive the energy density \mathcal{E} as a function of the density ρ and the renormalized coupling g to two loops.

4.1 Mean-field energy density

Using equations (19, 30) the density in the mean-field approximation is

$$\rho_0(\mu) = \frac{\mu}{g}. \quad (52)$$

The chemical potential is obtained by inverting equation (52):

$$\mu_0(\rho) = g\rho. \quad (53)$$

Using equations (20, 52), the energy density in the mean-field approximation is given by

$$\mathcal{E}_0(\rho) = \frac{1}{2}g\rho^2. \quad (54)$$

4.2 One-loop energy density

Using equations (19, 39) we obtain the density in the one-loop approximation

$$\rho_{0+1}(\mu) = \frac{\mu}{g} - \frac{\mu}{8\pi} \left[L + 1 \right]. \quad (55)$$

Inverting equation (55) to obtain μ as a function of ρ , one finds

$$\mu_{0+1}(\rho) = g\rho + \frac{g^2\rho}{8\pi} \left[\bar{L} + 1 \right], \quad (56)$$

where $\bar{L} = \ln(g\rho/2M^2)$ and $g = g(M)$. Using equations (20, 39) and (56), the energy density in the one-loop approximation becomes

$$\mathcal{E}_{0+1}(\rho) = \frac{1}{2}g\rho^2 + \frac{g^2\rho^2}{16\pi} \left[\bar{L} + \frac{1}{2} \right]. \quad (57)$$

This agrees with the result obtained by Lozano [25], and by Haugset and Ravndal [26].

4.3 Two-loop energy density

Using equations (19, 49), we obtain the density in the two-loop approximation:

$$\rho_{0+1+2}(\mu) = \frac{\mu}{g} - \frac{\mu}{8\pi} \left[L + 1 \right] + \frac{g\mu}{32\pi^2} [1 + C]. \quad (58)$$

Inverting equation (58), we obtain the two-loop expression for the chemical potential:

$$\begin{aligned} \mu_{0+1+2}(\rho) &= g\rho + \frac{g^2\rho}{8\pi} \left[\bar{L} + 1 \right] \\ &\quad + \frac{g^3\rho}{64\pi^2} \left[\bar{L}^2 + 3\bar{L} - 2C \right]. \end{aligned} \quad (59)$$

The final result for the two-loop energy density then becomes

$$\begin{aligned} \mathcal{E}_{0+1+2}(\rho) &= \frac{1}{2}g\rho^2 + \frac{g^2}{16\pi}\rho^2 \left[\bar{L} + \frac{1}{2} \right] \\ &\quad + \frac{g^3}{128\pi^2}\rho^2 \left[\bar{L}^2 + 2\bar{L} - 1 - 2C \right]. \end{aligned} \quad (60)$$

Using the renormalization group equation (51) for the running coupling constant, we see that the two-loop results for the density, chemical potential, and energy density are independent of the renormalization scale M up to corrections of order g^4 .

5 Renormalization group

The two-loop results for the pressure and density include a logarithm L . The renormalization group can be used to

absorb this logarithm. The solution to the renormalization group equation (51) is

$$g(M_0) = \frac{g(M)}{1 - \frac{g(M)}{8\pi} \ln\left(\frac{M_0^2}{M^2}\right)}. \quad (61)$$

By substituting the running coupling constant into the expressions for the pressure and density in the mean-field approximation and choosing the renormalization scale $M_0 = \sqrt{\mu/2}$, we absorb this logarithm. In a similar manner, we can sum logs of the form $g^{n+1}L^n$ ($n = 1, 2, 3, \dots$) by using the running coupling in the remaining terms in expressions for the pressure and density. These terms are generated by expanding out equation (61) in powers of L . The result is

$$\rho(\mu) = \frac{\mu}{g_0} - \frac{\mu}{8\pi} + \frac{g_0\mu}{32\pi^2} [1 + C], \quad (62)$$

$$\mathcal{P}(\mu) = \frac{\mu^2}{2g_0} - \frac{\mu^2}{32\pi} + \frac{g_0\mu}{64\pi^2} [1 + C], \quad (63)$$

where $g_0 \equiv g(M_0)$. In the same way, we can sum leading (of the form $g^{n+1}\bar{L}^n$, where $n = 1, 2, 3, \dots$) and subleading logs (of the form $g^{n+m}\bar{L}^n$, where $n = 1, 2, 3, \dots$, and $m = 2, 3, \dots$) from all orders of perturbation theory in the chemical potential and energy density by using a running coupling constant $g(M_0)$ and choosing the renormalization scale $M_0 = \sqrt{\rho/2}$. These terms are generated by expanding out equation (61). The result is

$$\begin{aligned} \mu(\rho) = & g_0\rho + \frac{g_0^2\rho}{8\pi} [\ln(g_0) + 1] \\ & + \frac{g_0^3\rho}{64\pi^2} [\ln^2(g_0) + 3\ln(g_0) - 2C], \end{aligned} \quad (64)$$

$$\begin{aligned} \mathcal{E}(\rho) = & \frac{1}{2}g_0\rho + \frac{g_0^2\rho}{16\pi} \left[\ln(g_0) + \frac{1}{2}\right] \\ & + \frac{g_0^3\rho}{128\pi^2} [\ln^2(g_0) + 2\ln(g_0) - 1 - 2C], \end{aligned} \quad (65)$$

where $g_0 \equiv g(M_0)$.

We have approximated the two-body interaction potential by local interactions. In real systems, however, the interactions have a finite range a and so $1/a$ provides a natural ultraviolet cutoff M . In the dilute gas limit, we have $|\ln(\mu a^2/2)| \gg 1$ and $|\ln(\rho a^2/2)| \gg 1$ [10–15], and the running coupling constant is approximately equal to either $-8\pi[\ln(\mu a^2/2)]^{-1}$ or $-8\pi[\ln(\rho a^2/2)]^{-1}$, which essentially is the two-body T -matrix. By substituting these expressions for the coupling constant into the expressions for the pressure, density, chemical potential,

and energy density, we obtain the renormalization group improved results

$$\mathcal{P}(\mu) = -\frac{\mu^2 L}{16\pi} \left[1 + \frac{1}{2}L^{-1} + 2(1+C)L^{-2}\right], \quad (66)$$

$$\rho(\mu) = -\frac{\mu L}{8\pi} \left[1 + L^{-1} + 2(1+C)L^{-2}\right], \quad (67)$$

$$\begin{aligned} \mu(\rho) = & -\frac{8\pi\rho}{\bar{L}} \left[1 + \left(\ln[-\bar{L}/8\pi]\bar{L} - 1\right)\bar{L}^{-1}\right. \\ & \left. + \ln^2(\ln[-\bar{L}/8\pi])\bar{L}^{-2} - 3(\ln[-\bar{L}/8\pi])\bar{L}^{-2}\right. \\ & \left. - 2C\bar{L}^{-2}\right], \end{aligned} \quad (68)$$

$$\begin{aligned} \mathcal{E}(\rho) = & -\frac{4\pi\rho^2}{\bar{L}} \left[1 + \left(\ln[-\bar{L}/8\pi] - \frac{1}{2}\right)\bar{L}^{-1}\right. \\ & \left. + \ln^2(\ln[-\bar{L}/8\pi])\bar{L}^{-2} - 2(\ln[-\bar{L}/8\pi])\bar{L}^{-2}\right. \\ & \left. - (1+2C)\bar{L}^{-2}\right], \end{aligned} \quad (69)$$

where $L = [\ln(\mu a^2/2)]$ and $\bar{L} = [\ln(\rho a^2/2)]$. The full two-loop renormalization group improved results equations (66–69) are the main result of the present paper. The leading order results were first obtained by Schick [10], while leading corrections were considered in references [11–16].

We note that the pressure \mathcal{P} and the density ρ are expansions in powers of the two-body T -matrix, while the chemical potential μ and energy density \mathcal{E} are expansions in powers of $T^{2B} \ln(T^{2B})$ rather than T^{2B} . This difference can be traced back to the fact the quantum loop expansion for \mathcal{P} and ρ are expansions in powers of g , while μ and \mathcal{E} are expansions in powers of $g \ln(g)$.

6 Summary

In the present paper, we have studied a two-dimensional interacting homogeneous Bose gas at zero temperature. We have calculated the ground state pressure and energy density to second order in the quantum loop expansion. The results are independent of the arbitrary renormalization scale M .

We have applied the renormalization group to sum up leading and subleading logarithms from all orders in perturbation theory. In the dilute limit, the renormalization group improved pressure and energy density are expansions in powers of T^{2B} and $T^{2B} \ln(T^{2B})$, respectively, where T^{2B} is the two-body T -matrix. We have obtained the ground state pressure, density, chemical potential, and energy density to next-to-next-to-leading order.

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Appendix A: Formulas

The loop integrals that appear in our calculations involve integrations over the energy ω and the spatial momentum \mathbf{p} . The energy integrals are evaluated using contour integration.

The specific one-loop integral needed is

$$\int \frac{d\omega}{2\pi} \frac{1}{[\omega^2 - \epsilon^2(p) + i\epsilon]} = -\frac{i}{2\epsilon(p)}. \quad (\text{A.1})$$

The specific two-loop integrals needed are

$$\begin{aligned} & \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \frac{1}{[\omega_1^2 - \epsilon^2(p) + i\epsilon]} \\ & \times \frac{1}{[\omega_2^2 - \epsilon^2(q) + i\epsilon][(\omega_1 + \omega_2)^2 - \epsilon^2(r) + i\epsilon]} = \\ & \frac{1}{4\epsilon(p)\epsilon(q)\epsilon(r)[\epsilon(p) + \epsilon(q) + \epsilon(r)]}, \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} & \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \frac{\omega_1\omega_2}{[\omega_1^2 - \epsilon^2(p) + i\epsilon]} \\ & \times \frac{1}{[\omega_2^2 - \epsilon^2(q) + i\epsilon][(\omega_1 + \omega_2)^2 - \epsilon^2(r) + i\epsilon]} = \\ & \frac{1}{4\epsilon(r)[\epsilon(p) + \epsilon(q) + \epsilon(r)]}. \end{aligned} \quad (\text{A.3})$$

Here $r = |\mathbf{p} + \mathbf{q}|$.

Some of the one-loop momentum integrals are infrared divergent or ultraviolet divergent or both. They can be written in terms of the integral $I_{m,n}$, which is defined by

$$I_{m,n} = \left(\frac{e^\gamma M^2}{4\pi}\right)^\epsilon \int \frac{d^d p}{(2\pi)^d} \frac{p^{2m}}{p^n (p^2 + \Lambda^2)^{n/2}}. \quad (\text{A.4})$$

Here, M is a renormalization scale that ensures that $I_{m,n}$ has the canonical dimension also for $d \neq 2$. $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. With dimensional regularization, $I_{m,n}$ is given by the formula

$$\begin{aligned} I_{m,n} &= \frac{\Omega_d}{(2\pi)^d} \left(\frac{e^\gamma M^2}{4\pi}\right)^\epsilon \Lambda^{d+2m-2n} \\ & \times \frac{\Gamma\left(\frac{d-n}{2} + m\right) \Gamma\left(n - m - \frac{d}{2}\right)}{2\Gamma\left(\frac{n}{2}\right)}, \end{aligned} \quad (\text{A.5})$$

where $\Omega_d = 2\pi^{d/2}/\Gamma[d/2]$ is the area of the d -dimensional sphere.

The integrals $I_{m,n}$ satisfy the relations

$$\frac{d}{d\Lambda^2} I_{m,n} = -\frac{n}{2} I_{m+1,n+2}, \quad (\text{A.6})$$

$$(d + 2m - n) I_{m,n} = n I_{m+2,n+2}, \quad (\text{A.7})$$

$$\Lambda^2 I_{m,n} = I_{m-1,n-2} - I_{m+1,n}. \quad (\text{A.8})$$

The first relation follows directly from the definition of $I_{m,n}$. The second relation follows from integration by parts, while the last is simply an algebraic relation.

In two dimensions, these integrals have logarithmic and power ultraviolet divergences. The power divergences are set to zero in dimensional regularization, while the logarithmic divergences appear as poles in ϵ . The specific integrals are

$$\begin{aligned} I_{0,-1} &= -\frac{\Lambda^4}{32\pi} \left\{ \frac{1}{\epsilon} - L - \frac{1}{2} \right. \\ & \left. + \frac{1}{2} \left[L^2 + L - \frac{5}{2} + \frac{\pi^2}{2} \right] \epsilon + \mathcal{O}(\epsilon^2) \right\}, \end{aligned} \quad (\text{A.9})$$

$$\begin{aligned} I_{-1,-1} &= \frac{\Lambda^2}{8\pi} \left\{ \frac{1}{\epsilon} - L + 1 \right. \\ & \left. + \frac{1}{2} \left[(L-1)^2 + 1 + \frac{\pi^2}{2} \right] \epsilon + \mathcal{O}(\epsilon^2) \right\}, \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} I_{1,1} &= -\frac{\Lambda^2}{8\pi} \left\{ \frac{1}{\epsilon} - L - 1 \right. \\ & \left. + \frac{1}{2} \left[(L+1)^2 - 4 + \frac{\pi^2}{2} \right] \epsilon + \mathcal{O}(\epsilon^2) \right\}, \end{aligned} \quad (\text{A.11})$$

$$I_{0,1} = \frac{1}{4\pi} \left\{ \frac{1}{\epsilon} - L + \frac{1}{2} \left[L^2 + \frac{\pi^2}{2} \right] \epsilon + \mathcal{O}(\epsilon^2) \right\}, \quad (\text{A.12})$$

where $L = \ln(\Lambda^2/4M^2)$.

The two-loop integrals needed can be expressed in terms of $J_{l,m,n}$

$$\begin{aligned} J_{l,m,n} &= \left(\frac{e^\gamma M^2}{4\pi}\right)^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \\ & \times \frac{[p/\sqrt{p^2 + \Lambda^2}]^l [q/\sqrt{q^2 + \Lambda^2}]^m [r/\sqrt{r^2 + \Lambda^2}]^n}{p\sqrt{p^2 + \Lambda^2} + q\sqrt{q^2 + \Lambda^2} + r\sqrt{r^2 + \Lambda^2}}. \end{aligned} \quad (\text{A.13})$$

In two dimensions, these integrals have quadratic and double logarithmic divergences that cancel in the particular combination in equation (43), leaving us with a logarithmically divergent integral. We write the integral as

$$J = J_{\text{div}} + J_{\text{num}}, \quad (\text{A.14})$$

where the ultraviolet divergence of the integral J has been isolated:

$$\begin{aligned} J_{\text{div}} &= 2 \left(\frac{e^\gamma M^2}{4\pi}\right)^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \left[2 - \frac{p}{\sqrt{p^2 + \Lambda^2}} \right. \\ & \left. - \frac{\sqrt{p^2 + \Lambda^2}}{p} \right] \int \frac{d^d q}{(2\pi)^d} \frac{1}{q\sqrt{q^2 + \Lambda^2}}. \end{aligned} \quad (\text{A.15})$$

The first term inside the square brackets in equation (A.15) has only a power divergence, and so it is set to zero in dimensional regularization. In terms of $I_{m,n}$, the remaining terms can be written as

$$J_{\text{div}} = -2I_{0,1} [I_{1,1} + I_{-1,-1}]. \quad (\text{A.16})$$

The remaining finite part of the integral J can be evaluated directly in two dimensions and reads

$$J_{\text{num}} = \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \left\{ \left[\frac{6p}{\sqrt{p^2 + \Lambda^2}} - \frac{2}{p} \sqrt{p^2 + \Lambda^2} \right. \right. \\ \left. \left. - \frac{3pqr}{\sqrt{p^2 + \Lambda^2} \sqrt{q^2 + \Lambda^2} \sqrt{r^2 + \Lambda^2}} \right. \right. \\ \left. \left. - \frac{r\sqrt{p^2 + \Lambda^2} \sqrt{q^2 + \Lambda^2}}{pq\sqrt{r^2 + \Lambda^2}} \right] \right. \\ \left. \times \frac{1}{p\sqrt{p^2 + \Lambda^2} + q\sqrt{q^2 + \Lambda^2} + r\sqrt{r^2 + \Lambda^2}} \right. \\ \left. - \frac{2}{q\sqrt{q^2 + \Lambda^2}} \left[2 - \frac{p}{\sqrt{p^2 + \Lambda^2}} - \frac{\sqrt{p^2 + \Lambda^2}}{p} \right] \right\}.$$

Since the only scale in the integrand in equation (A.17) is Λ , it follows from dimensional analysis that J_{num} is proportional to Λ^2 . The numerical value is

$$J_{\text{num}} = -1.76 \times 10^{-5} \Lambda^2. \quad (\text{A.17})$$

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